Free-surface oscillations in a slowly rotating liquid

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Free-surface oscillations of a liquid relative to an equilibrium state of uniform rotation about the vertical of an axisymmetric container are considered for small $\alpha = \omega^2 a/g$, where ω is the angular velocity of rotation and a the cylinder radius. A variational approximation is used to obtain explicit results, with an error of $O(\alpha^2)$, for axisymmetric gravity and inertial waves in a flat-bottomed circular cylinder; these results are found to be in agreement with observations reported by Fultz (1962). The first-order (in ω) effects of rotation on asymmetric waves in a circular cylinder also are determined.

1. Introduction

We consider small, free-surface oscillations of a liquid relative to an equilibrium state of uniform rotation about the vertical (z-axis) with angular velocity ω. We shall assume that the container is a surface of revolution, say

$$z/d = f(r/a), \tag{1.1}$$

where d is a typical depth, a a typical radius, and z and r cylindrical polar coordinates; in the particular case of a flat-bottomed circular cylinder, we shall choose d as the mean depth and a as the radius (so that the volume is $\pi a^2 d$). The equilibrium free-surface then is given by

$$z = z_0(r) = z_0(0) + \tfrac{1}{2}(\omega^2/g)\,r^2, \tag{1.2}$$

where $z_0(0)$ is determined by the constraint of constant volume. This yields

$$z_0(0) = d - \tfrac{1}{4}(\omega^2/g)\,a^2 \tag{1.3}$$

for a circular cylinder with bottom at $z = 0$. We shall impose the restriction $z_0(0) > 0$.

Two types of oscillations are possible, namely gravity waves and inertial waves, the frequencies of which are $O(g^{\frac{1}{2}})$ and $O(\omega)$ as $\omega \to 0$. The inertial waves tend to purely internal motions, unaccompanied by free-surface displacement, in this limit. Let

$$\delta = d/a \tag{1.4}$$

be a depth parameter and

$$\alpha = \omega^2 a/g = z_0'(r)\big|_{r=a} \tag{1.5}$$

a measure of the rotational effects; then, at least in so far as δ is not too small, appropriate, dimensionless forms for the angular frequencies are

$$\sigma_g = (g/a)^{\frac{1}{2}}\,G(\alpha,\delta), \quad \sigma_\omega = \omega\Omega(\alpha,\delta). \tag{1.6a,b}$$

If $\alpha \ll \delta \ll 1$, $(gd/a^2)^{\frac{1}{2}}$ would be a more convenient reference for σ_g; if $\delta = O(\alpha)$

the distinction between gravity and inertial waves becomes blurred; the domain $\delta \ll \alpha$ is excluded by the restriction $z_0(0) > 0$, which implies $\alpha < 4\delta$ for the circular cylinder.

Both types of oscillations have been considered previously by Lamb (1932, §§206–12), following earlier work by Kelvin. Lamb based his analysis on the shallow-water approximation ($\delta \ll 1$) and the approximation of the mean free surface by a plane (*planar approximation*). We remark that inertial oscillations do not appear in the shallow-water formulation for a liquid of *uniform* depth. Thus Lamb's analysis for the circular cylinder yields only two modes of oscillation (both of which are gravity waves in the present sense) for each pair of radial and azimuthal wave numbers, whereas his analysis for a paraboloid yields three modes, of which one is an inertial oscillation. Miles (1959) extended Lamb's formulation by removing the shallow-water, but not the planar, approximation. The presumption in these analyses is that the planar approximation should be valid as $\alpha \to 0$.

Fultz (1962) has remarked that the planar approximation is necessarily inconsistent for axisymmetric gravity waves† in the sense that both the rotation-induced shift in σ_g and the free-surface slope are of the same order of magnitude —namely $O(\alpha)$ as $\alpha \to 0$. Fultz was led to this conclusion by the disagreement between his own experimental results and Lamb's theoretical result

$$\sigma_g^2 = \sigma_0^2 + 4\omega^2 \quad (\delta \to 0), \tag{1.7}$$

where σ_0 denotes the frequency for $\omega = 0$. Subsequently, Platzman (1962) carried out a calculation including free-surface slope and found

$$\frac{\sigma^2 - \sigma_0^2}{4\omega^2} \to 0{\cdot}69 \left(\frac{\alpha}{\delta} \to 0,\ \delta \to 0\right), \tag{1.8}$$

in agreement with experiment.

Murty (1962) has since carried out a rather extensive analysis for axisymmetric, shallow-water oscillations in a cylindrical tank with a paraboloidal bottom. Miles & Ball (1963) have considered both axisymmetric and asymmetric shallow-water oscillations in a paraboloid and have obtained finite-amplitude solutions for two important (axisymmetric and asymmetric) modes.

We present here a corrected (*vis-à-vis* Miles 1959) formulation for the general problem (§2 below) and a variational integral of this formulation (§3). We shall use this result primarily to obtain (§4) explicit approximations, with an error of $O(\alpha^2)$, to both σ_g and σ_ω for axisymmetric oscillations in a flat-bottomed circular cylinder.

We also shall present approximations, accurate through terms of $O(\omega)$, to both σ_g and σ_ω for asymmetric waves in a circular cylinder (the planar approximation is, of course, consistent with these results). The approximation to σ_g constitutes a generalization of previous results of Rayleigh (1903) and Miles (1959) for the limiting cases $\delta \to 0$ and ∞, respectively. The approximation for σ_ω, which appears to be new, is presented in implicit form for arbitrary δ and in explicit expansions for either small or large δ.

† The writer had been led to this same conclusion in considering the Cauchy–Poisson problem for a rotating liquid (Miles 1963).

Finally, we remark that the numerical results for asymmetric inertial waves given in Table 2 of Miles (1959) are valid only for $\alpha \to 0$. More importantly, the limit point (of eigenvalues) encountered there was a spurious consequence of the inconsistent formulation.

2. Formulation

Let $\{r, \theta, z\}$ be cylindrical polar co-ordinates in a reference frame that rotates with the tank. Assuming small, inviscid disturbances of angular frequency σ and azimuthal wave-number m, we may derive the perturbation pressure $p - p_0$ and the perturbation velocity $\{u, v, w\}$ from the acceleration potential χ according to

$$\rho^{-1}(p-p_0)+g(z-z_0) = \chi(r,\theta,z,t) = \phi(r,z)\, e^{i(\sigma t+m\theta)}, \tag{2.1}$$

(where ρ is the liquid density) and

$$\{u,v,w\} = \frac{i}{\sigma}\left\{(1-\mu^2)^{-1}\left(\frac{\partial}{\partial r}+\frac{\mu m}{r}\right),\quad i(1-\mu^2)^{-1}\left(\mu\frac{\partial}{\partial r}+\frac{m}{r}\right),\ \frac{\partial}{\partial z}\right\}\chi, \tag{2.2}$$

where

$$r^{-1}(r\phi_r)_r-(m/r)^2\phi+(1-\mu^2)\,\phi_{zz} = 0, \tag{2.3}$$

and

$$\mu = 2\omega/\sigma. \tag{2.4}$$

We can restrict m and ω to be positive without loss of generality.

The kinematic boundary condition at the surface of (1.1) is

$$w = \delta f' u \quad (z = df). \tag{2.5}$$

The linearized boundary conditions at the free surface $z = z_0(r)+\zeta(r,\theta,t)$, where $p = p_0$, are

$$\chi = g\zeta \quad \text{and} \quad w = \frac{Dz}{Dt} \doteqdot i\sigma\zeta + z_0' u \quad (z = z_0+\zeta \doteqdot z_0). \tag{2.6a, b}$$

Eliminating ζ between (2.6a, b) and substituting u and w from (2.2) into (2.5) and (2.6b) and $z_0(r)$ from (1.2) into (2.6b), we obtain

$$\phi_z-\delta(1-\mu^2)^{-1}f'(\phi_r+\mu m r^{-1}\phi) = 0 \quad (z = df(r/a)) \tag{2.7}$$

and

$$\sigma^2\phi-g\phi_z+\omega^2(1-\mu^2)^{-1}(r\phi_r+\mu m\phi) = 0 \quad (z = z_0(r)). \tag{2.8}$$

The derivation of (2.1)–(2.6a) is given in Miles (1959), but the term $z_0'u$ in (2.6b) was neglected, and the free-surface boundary condition given there contained only the first two terms in (2.8).

We now have the following eigenvalue problem: given the container shape $f(r/a)$ and values of the parameters α and δ, find those values of σ—or, equivalently, μ—for which there exist non-trivial solutions to the partial differential equation (2.3) and the boundary conditions (2.7) and (2.8).

3. Variational approximation

Multiplying (2.3) by ϕ, integrating over a meridian cross-section of the liquid, and invoking (2.7) and (2.8) after appropriate partial integrations, we obtain the variational integral

$$\Phi = (\sigma^2/g)\,(1-\mu^2+\tfrac{1}{4}\mu^3 m)\int_{z=z_0(r)}\phi^2 r\,dr - \mu m\int_{z=df(r/a)}\phi^2 dz - \iint[\phi_r^2+(m/r)^2\phi^2+(1-\mu^2)\,\phi_z^2]\,r\,dr\,dz. \tag{3.1}$$

The first integral is over a meridian segment of the free surface, the second over a meridian segment of the container, and the third over a meridian cross-section of the liquid; the respective limits of integration for the circular cylinder are $(0, a)$, $(0, z_0(a))$, and $(0, a;\ 0, z_0(r))$. We may show, by conventional techniques, that the integral Φ is stationary with respect to first-order variations of ϕ about the true solution to (2.3), subject to the boundary conditions (2.7) and (2.8).

We shall find it convenient to pose trial solutions that satisfy the differential equation (2.3) exactly and to invoke the variational principle only in respect to the boundary conditions (2.7) and (2.8). We then may transform (3.1) to

$$\Phi = \left(\frac{1-\mu^2}{g}\right)\int_{z=z_0(r)}\left\{\sigma^2\phi - g\phi_z + \left(\frac{\omega^2}{1-\mu^2}\right)(r\phi_r + \mu m\phi)\right\}\phi r\,dr + \int_{z=df(r/a)} \phi\{(1-\mu^2)\,\phi_z r\,dr - (r\phi_r + \mu m\phi)\,dz\}. \quad (3.2)$$

In the particular case of the circular cylinder, we may rewrite the last integral in (3.2) according to

$$\int_{z=df(r/a)} \ldots = (1-\mu^2)\int_0^a (\phi\phi_z)_{z=0}\, r\,dr - \int_0^{z_0(a)} (r\phi\phi_r + \mu m\phi^2)_{r=a}\,dz. \quad (3.3)$$

4. Axisymmetric waves in a circular cylinder

We turn now to the special case of axisymmetric ($m = 0$) waves in a circular cylinder. A suitable trial solution, which satisfies the differential equation (2.3) and the boundary condition (2.7) at both $r = a$ and $z = 0$, is

$$\phi = \sum_k A_k\phi_k, \quad \phi_k = a^{-\frac{1}{2}} J_0(kr/a)\cosh(\kappa z/a), \quad (4.1a, b)$$

where the summation is over the roots of

$$J_1(k) = 0, \quad (4.2)$$

$$\kappa = (1-\mu^2)^{-\frac{1}{2}}\, k, \quad (4.3)$$

and we have included the factor $a^{-\frac{1}{2}}$ for dimensional convenience. We remark that the ϕ_k form a complete set for the domain $0 < z < z_0(r)$, $0 < r < a$ and that the members of this set are orthogonal in, and only in, the limit $\alpha \to 0$.

Substituting (4.1) into (3.2) and requiring Φ to be stationary with respect to independent variations of each of the A_k would yield a set of linear equations in the A_k; requiring the determinant of these equations to vanish then would yield the eigenvalue equation for σ. Such a procedure permits the eigenvalues to be determined to any desired accuracy, but we shall restrict the subsequent development to the simplest approximation of this type, namely, that obtained by retaining only a single term in the expansion.

Let Φ_k be that approximation to Φ obtained by substituting $\phi = \phi_k$ from (4.1*b*) into (3.2), namely

$$\Phi_k = \left(\frac{1-\mu^2}{g}\right)\int_0^a \left\{\sigma^2\phi_k - g\phi_{kz} + \left(\frac{\omega^2}{1-\mu^2}\right) r\phi_{kr}\right\}\phi_k r\,dr \tag{4.4a}$$

$$= \frac{1}{a^2}\int_0^a \left\{(1-\mu^2)\left[\left(\frac{\sigma^2 a}{g}\right) - \kappa\tanh\left(\frac{\kappa z_0}{a}\right)\right] J_0^2\left(\frac{kr}{a}\right) + \alpha\left(\frac{kr}{a}\right) J_0'\left(\frac{kr}{a}\right) J_0\left(\frac{kr}{a}\right)\right\}\cosh^2\left(\frac{\kappa z_0}{a}\right) r\,dr. \tag{4.4b}$$

Remarking that the solution $\quad \phi = A_k\phi_k + O(\alpha) \qquad (4.5a)$

is capable of satisfying the free-surface boundary condition (2.8) as $\alpha \to 0$, requiring the corresponding approximation $\Phi = A_k\Phi_k$ to be stationary with respect to a first-order variation of A_k, and invoking the variational principle (which implies that the error in Φ at the stationary point must be of the order of the square of the error in ϕ), we obtain the eigenvalue equation†

$$0 = \Phi_k + O(\alpha^2). \tag{4.5b}$$

We therefore shall approximate Φ_k by expanding the integrand of (4.4*b*) in powers of α and neglecting terms of $O(\alpha^2)$. Substituting $z_0(r)$ from (1.2) and (1.3), expanding the hyperbolic functions about $z_0(r) = d$, and setting $r = ax$, we obtain

$$\Phi_k = \cosh^2(\kappa\delta)\int_0^1 \{(1-\mu^2)[(\sigma^2 a/g) - \kappa\tanh(\kappa\delta)]J_0^2(kx) + \alpha kx J_0(kx) J_0'(kx)\}\,x\,dx \; - \tfrac{1}{2}\alpha k^2\int_0^1 (x^2 - \tfrac{1}{2}) J_0^2(kx)\,x\,dx + O(\alpha^2). \tag{4.6}$$

Carrying out the integrations [see Erdélyi, Magnus, Oberhettinger & Tricomi, 1954, § 19.1 (39), for the last], invoking (4.2), and equating Φ_k to zero, we obtain

$$(\sigma^2 a/g) - \kappa\tanh(\kappa\delta) + \tfrac{1}{12}\alpha\kappa^2\,\mathrm{sech}^2(\kappa\delta) + O(\alpha^2) = 0. \tag{4.7}$$

As anticipated in (1.6), the frequencies given by (4.7) must be either $O(g/a)^{\frac{1}{2}}$ or $O(\omega)$ as $\alpha \to 0$.

Considering first gravity waves, we have the zero'th approximation

$$\sigma_g^2 = \sigma_0^2 + O(\omega^2), \quad \sigma_0^2 = (kg/a)\tanh(k\delta). \tag{4.8a, b}$$

It follows that $\mu = O(\alpha^{\frac{1}{2}})$ and hence that we may approximate κ according to

$$\kappa = k(1-\mu^2)^{-\frac{1}{2}} = k[1 + 2(\omega/\sigma_0)^2 + O(\alpha^2)]. \tag{4.9}$$

Substituting (4.9) into (4.7) we obtain

$$\sigma_g^2 = \sigma_0^2 + 2\omega^2[1 + 2k\delta\,\mathrm{csch}\,(2k\delta) - \tfrac{1}{24}k^2\,\mathrm{sech}^2(k\delta)] + O(\alpha^2 g/a). \tag{4.10}$$

† The determinant of the aforementioned linear equations in A_k would comprise the diagonal elements Φ_k and, in consequence of the orthogonality of the ϕ_k for $\alpha = 0$, off-diagonal elements of $O(\alpha)$. It therefore would be equal to the product of these diagonal elements plus terms of $O(\alpha^2)$, and requiring it to vanish would yield (4.5 *b*) without recourse to the variational principle.

The gravest mode, which may be expected to dominate observed motions, corresponds to the smallest root of (4.2), namely $k = 3{\cdot}832$. Substituting this value into (4.10), we obtain the following numerical results:

$\frac{d}{a} =$	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	∞
$\frac{\sigma^2-\sigma_0^2}{4\omega^2} =$	0·694	0·688	0·651	0·555	0·514	$\frac{1}{2}$

These are within the experimental scatter, for $\alpha/\delta < 0{\cdot}5$, of the observations reported by Fultz (1962, figure 12).

Turning to inertial waves, we have the zero'th approximations

$$\sigma_\omega^2 = 0, \ \tanh(\kappa_0\delta) = 0 \quad (\alpha \to 0), \tag{4.11a, b}$$

which imply

$$\kappa_0\delta = ip\pi \quad (p = 1, 2, \ldots) \tag{4.12a}$$

and

$$(2\omega/\sigma_\omega)^2 = 1 + (k\delta/p\pi)^2 \quad (\alpha \to 0). \tag{4.12b}$$

We may regard (4.12*b*) as a first approximation for the frequency in the sense that (4.11*a*) is the zero'th approximation. Substituting (4.12*a*, *b*) into the first and third terms of (4.7), we obtain the second approximation

$$(2\omega/\sigma_\omega)^2 = 1 + (k\delta_p)^2\left\{1 + \frac{\alpha}{\delta}\left[\frac{8\delta_p^2}{1+(k\delta_p)^2} - \frac{1}{6}\right] + O(\alpha^2)\right\}, \tag{4.13}$$

where

$$\delta_p = \delta/p\pi = d/p\pi a. \tag{4.14}$$

The observations of axisymmetric inertial waves reported by Fultz (1962, figure 4) were for rather small values of α/δ, and the difference between the approximations of (4.12*b*) and (4.13) is within the experimental scatter. It does appear, however, that the term of $O(\alpha/\delta)$ in (4.13) might be of interest in some geophysical applications.

We emphasize that the foregoing approximations break down for $\delta = O(\alpha)$ as $\alpha \to 0$, in which neighbourhood (cf. remarks following (1.6) above)

$$\sigma_0^2/\omega^2 \to k^2\delta/\alpha \quad (\delta \to 0). \tag{4.15}$$

This non-uniform validity in the neighbourhood of $\delta = 0$ is borne out by the divergence between (4.10) and the observations reported by Fultz (1962, figure 12) for non-small values of α/δ. We also recall that inertial waves degenerate to null motions as $\delta \to 0$ (cf. the remarks in third paragraph of § 1 above).

5. Asymmetric waves in a circular cylinder

We may generalize the analysis of the preceding section by introducing

$$\phi_{mk} = a^{-\frac{1}{2}}J_m(kr/a)\cosh(\kappa z/a), \tag{5.1}$$

in place of ϕ_k and choosing the k in such a way that the ϕ_{mk} constitute a complete set of functions for the domain $0 < z < z_0(r)$, $0 < r < a$; κ remains as in (4.3). The function ϕ_{mk} satisfies the differential equation (2.3) and the boundary condition (2.7) at $z = 0$, but it satisfies (2.7) at $r = a$ if and only if k is a zero of

$$F_m(k, \mu) = kJ'_m(k) + \mu m J_m(k). \tag{5.2}$$

The eigenfunctions given by (5.1) *and* the zeros of (5.2) form a complete, orthogonal set in the limit $\alpha = 0$, and a repetition of the argument given in the preceding section implies that the evanescence of Φ_{mk} (obtained by approximating ϕ by the single term ϕ_{mk} in Φ and choosing k as a zero of F_m) yields the corresponding eigenvalue with an error of $O(\alpha^2)$. If we were to include N members of this set in the expansion of ϕ, however, we should be faced with the solution of $N+1$ simultaneous equations in μ and $k_1, k_2, \ldots, k_N$, of which the first N would be given by the requirement that $k_1, \ldots, k_N$ be zeros of F_m and the last by the requirement that the determinant of the N variational constraints on $A_1, \ldots, A_N$ vanish. In fact, we shall obtain explicitly only such approximations as are sufficient to exhibit the first order (in ω) effects of rotation and shall invoke the variational principle only for gravity waves.

Considering first, and briefly, gravity waves, we start from the limiting results

$$J'_m(k) = 0, \quad \sigma^2 = \sigma_0^2 \equiv (kg/a)\tanh(k\delta) \quad (\alpha = 0). \tag{5.3a, b}$$

It follows from (1.5), (2.4) and (5.3b) that $\mu = O(\alpha^{\frac{1}{2}})$ as $\alpha \to 0$ and hence from (5.2) that the error in the value of k given by (5.3a) also is $O(\alpha^{\frac{1}{2}})$. We therefore may use this value of k to obtain a variational approximation to σ with an error of $O(\alpha)$.

Substituting ϕ_{mk} from (5.1) into (3.2) and (3.3), neglecting terms of $O(\alpha)$ and $O(\mu^2)$, carrying out the integrations, and invoking (5.3a), we obtain

$$\Phi_{mk} = \tfrac{1}{2}J_m^2(k)\cosh^2(k\delta)\{[(\sigma^2 a/g) - k\tanh(k\delta)][1-(m/k)^2] - \mu m[\delta\,\mathrm{sech}^2(k\delta) + k^{-1}\tanh(k\delta)]\} + O(\alpha). \tag{5.4}$$

Setting $\Phi_{mk} = 0$ and using (5.3b), we obtain

$$\sigma_g = \pm\sigma_0 + [1 + 2k\delta\,\mathrm{csch}(2k\delta)]\,m\,(k^2-m^2)^{-1}\omega + O(\alpha\sigma_0). \tag{5.5}$$

The result (5.5), which *is* consistent with the planar approximation, agrees with the results obtained previously by Rayleigh (1903) for $k\delta \to 0$ and by Miles (1959) for $k\delta \to \infty$. We emphasize, however, that the numerical results given in Table 1 (where $\beta = 2\alpha$) of the latter paper are valid only for small α in consequence of the planar approximation.

Turning to asymmetric inertial waves, we may start from the zero'th approximation of (4.11a, b), but now k has to be determined before proceeding to the first approximation. Substituting (4.12b) into (5.2) and equating the result to zero, we obtain the implicit relations

$$\lambda(k^2) = \mp[1+(k\delta_p)^2]^{-\frac{1}{2}} + O(\alpha) = -\sigma/2\omega, \tag{5.6a, b}$$

where δ_p is defined by (4.14),

$$\lambda(k^2) = \frac{mJ_m(k)}{kJ'_m(k)} \equiv \frac{J_{m-1}(k)+J_{m+1}(k)}{J_{m-1}(k)-J_{m+1}(k)}, \tag{5.7}$$

and the alternative signs in (5.6) *et seq.* are ordered.

The function $\lambda(k^2)$ [see Lamb, p. 323, where $\lambda \equiv -y$, or Miles (1959), figures 1 and 2, where $\lambda \equiv \lambda_2$] comprises an infinite sequence of monotonically increasing branches with infinite discontinuities at the zeros of J'_m. The first branch tends asymptotically to zero as $k^2 \to -\infty$ and passes through (0, 1); however, the intersection at $k = 0$ and $\lambda = 1$ yields only a trivial solution ($\sigma = -2\omega, u = v = w = 0$).

The remaining branches are located between the successive zeros of J'_m and pass through the points $(k^2_{m-1,n}, -1)$, $(k^2_{m,n}, 0)$ and $(k^2_{m+1,n}, +1)$, where $k_{m,n}$ denotes the nth zero of $J_m(k)$:

$$J_m(k_{m,n}) = 0 \quad (n = 1, 2, \ldots). \tag{5.8}$$

Designating the intersections of $\pm[1+(k\delta_p)^2]^{\frac{1}{2}}$ with $-\lambda(k^2)$ by $k_\pm$, we infer that

$$k^2_{m-1,n} < k^2_+ < k^2_{m,n} < k^2_- < k^2_{m+1,n}. \tag{5.9}$$

We also note that λ satisfies the Riccati equation

$$2z\lambda'(z) = (m^{-1}z - m)\,\lambda^2 + m. \tag{5.10}$$

We may solve (5.6) graphically, numerically (using, for example, the Newton–Raphson method), or through expansion in an appropriate parameter; we shall consider only the last approach. Considering first $\delta \to 0$, we may expand the right-hand side of (5.6a) about $\delta = 0$ and the left-hand side about $k^2_{m+1,n}$ to obtain

$$\mp 1 + \tfrac{1}{2}m^{-1}(k^2 - k^2_{m\mp1,n}) = \mp(1 - \tfrac{1}{2}\delta_p^2 k^2) + O(k\delta_p)^4, \tag{5.11}$$

which yields

$$k^2 = (1 \pm m\delta_p^2)\,k^2_{m\mp1,n} + O(k\delta_p)^4 \tag{5.12a}$$

and

$$\sigma_\omega/2\omega = \pm[1 - \tfrac{1}{2}\delta_p^2(1 \pm m\delta_p^2)\,k^2_{m\mp1\,n} + \tfrac{3}{8}(k_{m\mp1,n}\delta_p)^4 + O(k^4\delta_p^6)]. \tag{5.12b}$$

Turning to deep water, we may expand the right-hand side of (5.6a) about $k\delta = \infty$ and the left-hand side about $k^2_{m,n}$ to obtain

$$\tfrac{1}{2}(m/k^2_{m,n})(k^2 - k^2_{m,n}) = \mp(k_{m,n}\delta_p)^{-1} + O(k\delta_p)^{-3}, \tag{5.13}$$

which yields

$$k^2 = k^2_{m,n} \mp 2(m\delta_p)^{-1}k_{m,n} + O(k^{-1}\delta_p^{-3}) \tag{5.14a}$$

and

$$\sigma_\omega/2\omega = \pm(k_{m,n}\delta_p)^{-1} + m^{-1}(k_{m,n}\delta_p)^{-2} \pm \tfrac{1}{2}(3m^{-2} - 1)(k_{m,n}\delta_p)^{-3} + O(k_{m\,n}\delta_p)^{-4}. \tag{5.14b}$$

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